Eigenvalues and Eigenvectors

Properties of Eigenvalues and Eigenvectors

Definition: The trace of a matrix A, designated by tr(A), is the sum of the elements on the main diagonal.

Property 1: The sum of the eigenvalues of a matrix equals the trace of the matrix.

Property 2: A matrix is singular if and only if it has a zero eigenvalue.

Property 3: The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

Property 4: If λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{A}$ is invertible, then $\mathbf{1/\lambda}$ is an eigenvalue of matrix $\mathbf{A}^{-1}$.

Properties of Eigenvalues and Eigenvectors

Property 5: If λ is an eigenvalue of A then $k\lambda$ is an eigenvalue of kA where k is any arbitrary scalar.

Property 6: If λ is an eigenvalue of A then λk is an eigenvalue of Ak for any positive integer k.

Property 8: If λ is an eigenvalue of A then λ is an eigenvalue of AT.

Property 9: The product of the eigenvalues (counting multiplicity) of a matrix equals the determinant of the matrix.

Eigenvalues

Example 1: Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix}$$

$$= (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda + 1)(\lambda + 2)$$

two eigenvalues: $-1, -2$

Note: The roots of the characteristic equation can be repeated.

That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^3$$

$$= 0$$

λ = 2 is an eigenvaluer of multiplicity 3.

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

SOLUTION:

- find the **eigenvalues** of the matrix.

FINDING EIGENVALUES

- To do this, we find the values of λ which satisfy the **characteristic equation** of the matrix A, namely those values of λ for which

$$\det(A - \lambda I) = 0,$$

where I is the 3×3 **identity matrix.**

- Form the matrix $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix}.$$

Notice that this matrix is just equal to A with λ subtracted from each entry on the main diagonal.

- Calculate $\det(A - \lambda I)$:

$$\begin{aligned}
\det(A-\lambda I) &= (1-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} \\
&= (1-\lambda)\left((-5-\lambda)(4-\lambda)-(3)(-6)\right)+3(3(4-\lambda)-3\times 6)+3(3\times(-6)-(-5-\lambda)6) \\
&= (1-\lambda)(-20+5\lambda-4\lambda+\lambda^2+18)+3(12-3\lambda-18)+3(-18+30+6\lambda) \\
&= (1-\lambda)(-2+\lambda+\lambda^2)+3(-6-3\lambda)+3(12+6\lambda) \\
&= -2+\lambda+\lambda^2+2\lambda-\lambda^2-\lambda^3-18-9\lambda+36+18\lambda \\
&= 16+12\lambda-\lambda^3.
\end{aligned}$$

- Therefore

$$\det(A-\lambda I) = -\lambda^3+12\lambda+16.$$

REQUIRED: To find solutions to $\det(A-\lambda I)=0$ i.e., to solve

$$\lambda^3-12\lambda-16=0. \tag{1}$$

* Such solutions **divide** the **constant** term (-16). The list of possible integer solutions is

$$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16.$$

* Taking $\lambda = 4$, we find that $4^3 - 12.4 - 16 = 0$.
* Now factor out $\lambda - 4$:

$$(\lambda - 4)(\lambda^2 + 4\lambda + 4) = \lambda^3 - 12\lambda^2 + 16.$$

* Solving $\lambda^2 + 4\lambda + 4$ by formula[1] gives

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4.1.4}}{2} = \frac{-4 \pm 0}{2},$$

and so $\lambda = -2$ (a repeated root).

- Therefore, the eigenvalues of A are $\lambda = 4, -2$. ($\lambda = -2$ is a repeated root of the **characteristic equation**.)